# 9/19 Recitation Solutions

1. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find the general solutions to $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$. How do they compare?

   The RREF of $A$ is $\begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 3 & 0 \end{bmatrix}$. This gives the system

   $$\begin{aligned} x_1 - 3x_3 + 4x_4 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned}$$

   Rewriting the pivot variables in terms of the free variables $x_3$ and $x_4$ gives the following general solution to $A\mathbf{x} = \mathbf{0}$:

   $$\mathbf{x} = \begin{bmatrix} 3x_3 - 4x_4 \\ -3x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

   The RREF of $\left[\begin{array}{cccc|c} A & & & & \mathbf{b} \end{array}\right]$ is $\left[\begin{array}{cccc|c} 1 & 0 & -3 & 4 & -1 \\ 0 & 1 & 3 & 0 & 1 \end{array}\right]$. This gives the system

   $$\begin{aligned} x_1 - 3x_3 + 4x_4 &= -1 \\ x_2 + 3x_3 &= 1 \end{aligned}$$

   Solving for the free variables gives the general solution to $A\mathbf{x} = \mathbf{b}$:

   $$\mathbf{x} = \begin{bmatrix} -1 + 3x_3 - 4x_4 \\ 1 - 3x_3 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

   We can see that the solutions to $A\mathbf{x} = \mathbf{b}$ are obtained by adding the particular solution $\mathbf{x}_p = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ to the solutions of $A\mathbf{x} = \mathbf{0}$.

2. Suppose $A$ is an $m \times n$ matrix. Determine the number of potential solutions $A\mathbf{x} = \mathbf{b}$ can have if:

(a) $m \geq n$

First, if $\left[A \;\middle|\; \mathbf{b}\right]$ has a pivot in the last column, then there is no solution. Now assume that it does not have a pivot in the final (augmented) column. Since $m \geq n$, $A$ has at least as many rows as columns. This means that $A$ can have a pivot in every column, but not every row. If $A$ has a pivot in every column, then there is a unique solution, since there are no free variables. If $A$ doesn't have a pivot in every column, then there are infinitely many solutions, since there are free variables. So there can be 0, 1, or infinitely many solutions.

(b) $m < n$

First, if $\left[A \;\middle|\; \mathbf{b}\right]$ has a pivot in the last column, then there is no solution. Now assume that it doesn't have a pivot in the final (augmented) column. Since $m < n$, $A$ has more columns than rows. As a result, it is impossible for $A$ to have a pivot in every column. Hence there are necessarily free variables and, as a result, infinitely many solutions. Thus the system can have 0 or infinitely many solutions.

3. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Recall that the kernel of $T$ and the image of $T$ are the sets $\ker T = \{\mathbf{v} \in \mathbb{R}^n \,|\, T(\mathbf{v}) = \mathbf{0}\} \subseteq \mathbb{R}^n$ and $\text{im}T = \{T(\mathbf{v}) \,|\, \mathbf{v} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$.

(a) Show that $T(\mathbf{0}) = \mathbf{0}$.

Since $T$ is linear, we have that $T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0T(\mathbf{0}) = \mathbf{0}$.

(b) Show that the kernel of $T$ is a subspace of $\mathbb{R}^n$ using the definition of subspace.

- By part (a), we know that $T(\mathbf{0}) = \mathbf{0}$; hence $\mathbf{0} \in \ker T$.
- Let $\mathbf{v}, \mathbf{w} \in \ker T$. Then since $T$ is linear, $T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v})+T(\mathbf{w}) = \mathbf{0}+\mathbf{0} = \mathbf{0}$. Thus $\mathbf{v} + \mathbf{w} \in \ker T$.
- Let $\mathbf{v} \in \ker T$, $c \in \mathbb{R}$. Then since $T$ is linear, $T(c\mathbf{v}) = cT(\mathbf{v}) = c\mathbf{0} = \mathbf{0}$. Thus $c\mathbf{v} \in \ker T$.

(c) $T$ is called **onto** if $\text{im}T = \mathbb{R}^m$ and **one-to-one** if $\ker T = \{0\}$. Show that if $T$ is onto and one-to-one, then $T$ is invertible (Hint: show that the matrix associated to $T$ is invertible).

Let $A$ be the matrix associated to $T$. Recall that $\ker T = N(A)$ and $\text{im}T = C(A)$. Since $T$ is onto, we have that $C(A) = \mathbb{R}^m$. Thus $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^n$. As a result, $A$ must have a pivot in every row. Now since $T$ is one-to-one,

we have that $N(A) = \{\mathbf{0}\}$, which implies that there are no free variables and so there is a pivot in every column of $A$. Hence $A$ is an $n \times n$ matrix with $n$ pivots; that is, $A$ is invertible. It follows that $T$ is also invertible.

4. (a) Open the **Julia7SolutionSets** notebook to learn about a command that provides a spanning set of $N(A)$ and to see a simple function that provides the associated general solution.

   (b) **Coding Challenge**: Do coding challenge A in the notebook. At the bottom of the notebook, you can find a solution. As I am not a coder, my code may not be the most efficient or elegant. Feel free to improve on it!